Introduction to the Simply Typed Lambda Calculus

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Why type systems?

Many popular languages do not require tracking types of variables, which seems rather comfortable. Why should you care about type systems?

- Type checking prevents common and trivial bugs

  E.g. JavaScript or PHP: `1000 == "1e3"` is **true!**

  ⇒ weak + dynamic typing = have fun debugging!

- Working with types – "First think, then code"

  ⇒ cleaner, better organized results

- Types are never-outdated documentation!

- Type inference ⇒ not necesserily verbose

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Untyped Lambda Calculus

Some facts:

- fundamental formal system for computation
- introduced by Alonzo Church in 1936
- shown to be equivalent to Turing Machines in 1937
- used especially in type theory + PL research
- mother of all functional programming languages
  (especially ML and LISP family)

Syntax

Definition (Syntax of λ-calculus)

$\texttt{t} ::= \texttt{x}$	*variable*
$\lambda\texttt{x}.\ \texttt{t}$	*abstraction*
$\texttt{t}\ \texttt{t}$	*application*

- Abstraction = Function
- We will sometimes use parentheses
- When not, assume $\lambda\texttt{x}.\ \ldots = (\lambda\texttt{x}.\ \ldots)$
- $\lambda\texttt{x}.\ \texttt{x}\ \texttt{y}$: x is *bound*, y is *free*

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Evaluation

- one abstraction has exactly **one** parameter
- abstraction + application = *redex* (reducible expression)
- non-reducible terms = *values*
- in pure λ-calculus: abstraction is only kind of data

  $\Rightarrow$ computations always return other abstractions (only possible value!)
- *beta reduction*: one step of redex evaluation

  $(\lambda x.\, t_1)\, t_2 \to [x \mapsto t_2]t_1$

  $\Rightarrow$ function evaluation = term substitution
- We use *call-by-value* evaluation:

  evaluate terms left to right (depth-first),

  if remaining term is redex, recursively continue evaluating
- Other evaluation strategies also possible

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Naming of variables

What is wrong in the following evaluation?

$$[\mathrm{x} \mapsto \mathrm{z}](\lambda \mathrm{z}.\mathrm{x}) = (\lambda \mathrm{z}.[\mathrm{x} \mapsto \mathrm{z}]\mathrm{x}) = (\lambda \mathrm{z}.\mathrm{z})$$

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We changed the meaning from a constant function to an identity function!

Two possible solutions:

1. Allow substitution only if bound variable in abstraction not free in right-hand term of the substitution
2. Rename bound variable to unused name before applying such a substitution: $[\mathrm{x} \mapsto \mathrm{z}](\lambda \mathrm{z}.\mathrm{x}) = [\mathrm{x} \mapsto \mathrm{z}](\lambda \mathrm{y}.\mathrm{x}) = (\lambda \mathrm{y}.[\mathrm{x} \mapsto \mathrm{z}]\mathrm{x}) = (\lambda \mathrm{y}.\mathrm{z})$

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Currying and partial application

$$(\lambda x.\ \lambda y.\ x\ y)\ a\ b \to (\lambda y.\ a\ y)\ b \to a\ b$$

- Nested abstractions ‘simulate’ functions with multiple arguments
- Technique called *currying*, named after Haskell Curry
  (but thought to go back to Moses Schönfinkel)
- Inverse action – applying only some arguments to a curried function before e.g. passing it somewhere else is called *partial application*
- Here: Successive substitutions $[x \mapsto a]$ and $[y \mapsto b]$
  = passing first and second argument one after the other

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$$(\lambda \mathtt{x}.\ \lambda \mathtt{y}.\ \mathtt{x}\ \mathtt{y})\ \mathtt{a}\ \mathtt{b} \to (\lambda \mathtt{y}.\ \mathtt{a}\ \mathtt{y})\ \mathtt{b} \to \mathtt{a}\ \mathtt{b}$$

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Church booleans

Q: How can we calculate something meaningful, having only abstractions?

A: Find special abstractions we will treat as booleans ⇒ *Church booleans*

$$\mathtt{tru} = \lambda\mathtt{t}.\,\lambda\mathtt{f}.\,\mathtt{t}$$

$$\mathtt{fls} = \lambda\mathtt{t}.\,\lambda\mathtt{f}.\,\mathtt{f}$$

$$\mathtt{not} = \lambda\mathtt{b}.\,\mathtt{b}\ \mathtt{fls}\ \mathtt{tru}$$

$$\mathtt{and} = \lambda\mathtt{b}.\,\lambda\mathtt{c}.\,\mathtt{b}\ \mathtt{c}\ \mathtt{fls}$$

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$$\mathtt{test} = \lambda\mathtt{l}.\,\lambda\mathtt{m}.\,\lambda\mathtt{n}.\,\mathtt{l}\ \mathtt{m}\ \mathtt{n}$$

Problem: `test` always evaluates both arguments (= if-branches)

Solution: Wrap the arguments in dummy abstractions, unpack afterwards

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There is also an encoding for natural numbers, called *Church numerals*:

$$\mathtt{c}_0 = \lambda \mathtt{s}.\, \lambda \mathtt{z}.\, \mathtt{z} \qquad \mathtt{scc} = \lambda \mathtt{n}.\, \lambda \mathtt{s}.\, \lambda \mathtt{z}.\, \mathtt{s}\ (\mathtt{n}\ \mathtt{s}\ \mathtt{z})$$

$$\mathtt{c}_1 = \lambda \mathtt{s}.\, \lambda \mathtt{z}.\, \mathtt{s}\ \mathtt{z} \qquad \mathtt{plus} = \lambda \mathtt{m}.\, \lambda \mathtt{n}.\, \lambda \mathtt{s}.\, \lambda \mathtt{z}.\, \mathtt{m}\ \mathtt{s}\ (\mathtt{n}\ \mathtt{s}\ \mathtt{z})$$

$$\mathtt{c}_2 = \lambda \mathtt{s}.\, \lambda \mathtt{z}.\, \mathtt{s}\ (\mathtt{s}\ \mathtt{z}) \qquad \mathtt{times} = \lambda \mathtt{m}.\, \lambda \mathtt{n}.\, \mathtt{m}\ (\mathtt{plus}\ \mathtt{n})\ \mathtt{c}_0$$

$$\ldots \qquad \mathtt{iszro} = \lambda \mathtt{m}.\, \mathtt{m}\ (\lambda \mathtt{x}.\, \mathtt{fls})\ \mathtt{tru}$$

- Subtraction also possible, but more tricky
- With subtraction we also get equality:

$$\mathtt{eq} = \lambda \mathtt{n}.\, \lambda \mathtt{m}.\, \mathtt{and}\ (\mathtt{iszro}\ (\mathtt{minus}\ \mathtt{n}\ \mathtt{m}))\ (\mathtt{iszro}\ (\mathtt{minus}\ \mathtt{m}\ \mathtt{n}))$$

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$$c_2 = \lambda s.\ \lambda z.\ s\ (s\ z) \qquad times = \lambda m.\ \lambda n.\ m\ (plus\ n)\ c_0$$

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$$\mathtt{eq} = \lambda \mathtt{n}.\, \lambda \mathtt{m}.\, \mathtt{and}\ (\mathtt{iszro}\ (\mathtt{minus}\ \mathtt{n}\ \mathtt{m}))\ (\mathtt{iszro}\ (\mathtt{minus}\ \mathtt{m}\ \mathtt{n}))$$

Adding real booleans and numbers

No real programming language uses Church encoded data ⇒ inefficient!

Easy to extend syntax to support primitive data types as atomic values:

- Booleans: add `true`, `false`, `if t then t else t`
- Numbers: add `0`, `succ`, `pred`, `iszero`

Evaluation:

- `if`-condition evaluated ⇒ replace `if`-expression by correct branch
- `succ` + `pred` form redex ⇒ when they meet, remove
- `iszero 0` evaluates to `true`, otherwise `false`

Easy to convert between Church-encoded and primitive values, e.g.:

$$\texttt{realbool} = \lambda b.\ b\ \texttt{true}\ \texttt{false}$$

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- there are many possible extensions to the pure calculus:
- more primitive types, lists, tuples, recursion ($\rightarrow$ looping), ...
- either as part of formal definition or as *syntactic sugar*

  $\Rightarrow$ convenient notation for constructions that are possible,

  but are verbose/ugly/hard to use with base definition
- sugar helps keeping the core language clean and simple
- we do not add more stuff, finally move on to types ...

Motivation

Q: What about input like `if 0 then true else 0` or `succ false`?

A: Depending on the concrete expression and defined semantics:

- evaluation gets stuck at undefined state ($\rightarrow$ runtime-error)
- **worse**: evaluation continues, producing garbage, possibly undetected!

We need a way to easily and automatically check input **before** actual evaluation and only accept *well-typed* input that is playing by the rules!

Solution:

- Assign each function a type of the form $T_1 \rightarrow T_2$
- read: function taking value of type T_1, returning value of type T_2
- $\rightarrow$ = *type constructor*, T_n = *type variable*
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Syntax

Definition (Syntax of simply typed λ-calculus ($\lambda^{\rightarrow}$))

$\mathtt{t} ::=$	$\mathtt{x}$	*variable*
	$\lambda\mathtt{x}\colon \mathbf{T}.\ \mathtt{t}$	*abstraction*
	$\mathtt{t}$	*application*

- Invented by Church in 1940
- Only superficial difference: every abstraction gets type annotation
- *simply typed* = only way to construct types is $\rightarrow$

We need some rules for correct type annotation. Some new notation first:

- Let Γ be a set of assumptions about types of terms, e.g. free variables, called *typing context*
- $\Gamma \vdash \mathtt{t} : \mathtt{T}$ means 'under given assumptions the term t has the type T'
- Γ can be $\emptyset \Rightarrow$ can be omitted in that case: $\vdash \mathtt{t} : \mathtt{T}$ or $\mathtt{t} : \mathtt{T}$
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Typing rules

Definition (Typing of variables (T-Var))

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}$$

Definition (Typing of abstractions (T-Abs))

$$\frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1 .\, t_2 : T_1 \to T_2}$$

Definition (Typing of applications (T-App))

$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \quad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}$$

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Typing rules for booleans and numbers

Definition (T-True, T-False, T-If)

$$\texttt{true} : \texttt{Bool} \qquad \texttt{false} : \texttt{Bool} \qquad \frac{\Gamma \vdash \texttt{t}_1 : \texttt{Bool} \quad \Gamma \vdash \texttt{t}_2 : \texttt{T} \quad \Gamma \vdash \texttt{t}_3 : \texttt{T}}{\Gamma \vdash \texttt{if}\ \texttt{t}_1\ \texttt{then}\ \texttt{t}_2\ \texttt{else}\ \texttt{t}_3 : \texttt{T}}$$

Note that $\texttt{t}_2$ and $\texttt{t}_3$ in the if-expression must have the *same* type T!

Definition (T-Zero, T-Succ, T-Pred, T-IsZero)

$$0 : \texttt{Nat} \qquad \frac{\Gamma \vdash \texttt{t}_1 : \texttt{Nat}}{\Gamma \vdash \texttt{succ}\ \texttt{t}_1 : \texttt{Nat}} \qquad \frac{\Gamma \vdash \texttt{t}_1 : \texttt{Nat}}{\Gamma \vdash \texttt{pred}\ \texttt{t}_1 : \texttt{Nat}} \qquad \frac{\Gamma \vdash \texttt{t}_1 : \texttt{Nat}}{\Gamma \vdash \texttt{iszero}\ \texttt{t}_1 : \texttt{Bool}}$$

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Deduction example

Let's prove

$$\mathtt{f} : \mathtt{Bool} \to \mathtt{Bool} \vdash \lambda \mathtt{x} : \mathtt{Bool}.\, \mathtt{f}\,(\mathtt{if}\,\mathtt{x}\,\mathtt{then}\,\mathtt{false}\,\mathtt{else}\,\mathtt{x}) : \mathtt{Bool} \to \mathtt{Bool}$$

Proof.

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Properties of typing

Two important theorems can be shown for $\lambda^{\rightarrow}$ by structural induction:

Theorem (Progress)

Suppose $\mathtt{t}$ *is a closed, well-typed term (that is,* $\vdash \mathtt{t} : \mathtt{T}$ *for some* $\mathtt{T}$*). Then either* $\mathtt{t}$ *is a value or else there is some* $\mathtt{t}'$ *with* $\mathtt{t} \rightarrow \mathtt{t}'$*.*

Theorem (Preservation)

If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ *and* $\mathtt{t} \rightarrow \mathtt{t}'$*, then* $\Gamma \vdash \mathtt{t}' : \mathtt{T}$*.*

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Properties of typing

What does it mean?

- **Progress:**

  Every well-typed term can be reduced to a value
- **Preservation:**

  Every well-typed term evaluates to a well-typed term with the same type
- progress+preservation=***type safety***

  $\Rightarrow$ well-typed terms never get stuck during evaluation!

Properties of typing

- Another property that can be shown:

  type erasure does not influence evaluation $\Rightarrow$ Types can be (and are often!) removed during compilation, if everything is ok

- Reverse action – *type reconstruction*:

  finding a possible type of a term with incomplete type annotations

- if the reconstruction possible, the term is *typable*, if not:

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For each type we need to duplicate identical code with other type annotations!

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$\rightsquigarrow$ System F (Girard, 1972)

System F

Definition (Syntax of System F)

$\mathtt{t} ::=$	$\mathtt{x}$	*variable*
	$\lambda\mathtt{x} : \mathtt{T}.\, \mathtt{t}$	*abstraction*
	$\mathtt{t}$	*application*
	$\lambda\mathtt{X}.\, \mathtt{t}$	**type abstraction**
	$\mathtt{t}[\mathtt{T}]$	**type application**

uppercase letters = type variables, lowercase letters = terms

System F

- type abstraction/application works similar to normal, but we substitute type variables: $(\lambda X.\ t_{12})\ [T_2] \to [X \mapsto T_2]t_{12}$
- before annotated types had to be concrete, now they are abstracted
  $\Rightarrow$ we need new types and typing rules to express this
- type abstractions get a *universal type* of the form $\forall X.T$
- now we have two different type constructors: $\to$ and $\forall$
  $\Rightarrow$ *second-order lambda calculus*

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Rules for universal types

Definition (Typing of type abstractions (T-TAbs))

$$\frac{\Gamma, \mathtt{X} \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \lambda \mathtt{X}.\ \mathtt{t}_2 : \forall \mathtt{X}.\mathtt{T}_2}$$

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Examples

$\mathtt{id} = \lambda \mathtt{X}.\ \lambda \mathtt{x} : \mathtt{X}.\ \mathtt{x}$ type $\forall X.X \to X$

$\mathtt{idNat} = \mathtt{id}\ [\mathtt{Nat}] = \lambda \mathtt{x} : \mathtt{Nat}.\ \mathtt{x}$ type $Nat \to Nat$

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- As you see, parametric polymorphism is very expressive
- Haskell programs desugar to an ext. System F form during compilation
- preservation and progress theorems still hold in System F $\Rightarrow$ **type-safe**
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Type theory and OOP

- Important branch: λ-calculi with *subtyping* (Reynolds, Cardelli (1980's))
- Theoretical foundation of inheritance in OOP
- Extension: *Subtyping relation* with new set of deduction rules
- Says which types can be treated as more general types

  $\Rightarrow$ Functions can ignore specialisation and work on more inputs
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  isomorphism: types ≈ propositions, terms ≈ proofs!
  ⇒ Connection between constructive logic and computer science
- E.g. used for tools like *Coq* – interactive theorem prover
- Helps the user formulating assertions and finding proofs
- proof-checking = type-checking the program!
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Q: What do we get using type systems in the context of software verification?

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- our program will compile and execute **(catch syntax errors)**
- our functions will take and return the intended data types
  (catch violations of our mental model/the designed API)
- we can **control** which functions can do which **effects**,
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  (e.g. the Monad typeclass in Haskell)
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- more powerful type systems require more work by the developer
- a clear model of the types in an application is necessery
- types are also a form of formal specification –
  as usual, it is balance between more safety and more additional work
- λ-calculi are an important model to prove properties and develop new algorithms and abstractions
- results can be and are transferred to real-world programming languages
  $\Rightarrow$ developers profit from better, safer and more expressive tools :)

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  $\Rightarrow$ developers profit from better, safer and more expressive tools :)

Conclusion

- more powerful type systems require more work by the developer
- a clear model of the types in an application is necessery
- types are also a form of formal specification –
  as usual, it is balance between more safety and more additional work
- λ-calculi are an important model to prove properties and develop new algorithms and abstractions
- results can be and are transferred to real-world programming languages
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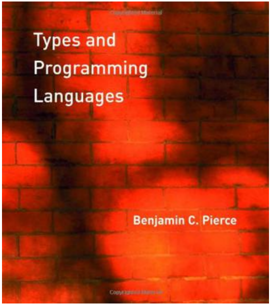

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